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SEVERAL PATTERNS OF FUNCTIONAL TRANSFORMATION AND GENERALIZED VARIATIONAL PRINCIPLES WITH SEVERAL ARBITRARY PARAMETERS

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Abstract—In this paper, the functional transformations of variational principles in elasticity are classified as three patterns: pattern I (relaxation pattern) is a generalized equivalent pattern in which constraint conditions are transformed into natural conditions; pattern II (augmented pattern) is a generalized equivalent pattern in which augmented conditions are transformed into natural conditions; pattern III (equivalent pattern) is a pattern in which a nonconditional functional is transformed into an equivalent functional with several arbitrary parameters. Pattern I is the well-known pattern of Lagrange multipliers method: patterns II and III are new patterns proposed in this paper. On the basis of pattern III, generalized variational principles with several arbitrary parameters are formulated, and the general and simple forms of the functional are defined. Many existing functionals of variational principles in elasticity are special cases of this functional.

1. INTRODUCTION

The variational principles in elasticity have been systematically discussed in [1–4]. In this paper, for the variational principles in elasticity, variables are classified as functional variables and augmented variables; conditions are classified as forced conditions which should be satisfied in advance by the functional variables, natural conditions (Euler equations and natural boundary conditions) and augmented conditions (the conditions or relations between the augmented and the functional variables, as well as that between various augmented variables). For instance, in the principle of potential energy the functional is the potential energy, the displacement $\mathbf{u}$ is the functional variable and both the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ are regarded as augmented variables. The displacement boundary conditions at the fixed boundary are the forced conditions which must be satisfied in advance by the functional variable $\mathbf{u}$. The differential equilibrium equations and the force boundary conditions at the free boundary both expressed in terms of displacement are the natural conditions derived from the stationary condition of potential energy, while the geometrical relations between the functional variable $\mathbf{u}$ and the augmented variable $\boldsymbol{\varepsilon}$, as well as the stress–strain relations between the augmented variables $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, are considered as augmented conditions.

When a statement of a variational principle is to be made, three aspects have to be mentioned: (1) which variables are chosen to be the functional variables; (2) which of the conditions are used as the forced conditions and which of them are the augmented conditions; (3) how to define the energy functional—the natural condition can be derived from stationary conditions of the functional.

The equivalent relation between two variational principles has been discussed frequently in literatures, but sometimes the meaning of the word equivalent are not exactly the same in different contexts. In order to have a clear and definite concept, in this paper three different cases of the equivalent relations are defined as follows:

(1) Two variational principles are said to be generalized equivalent if both the principles have the same set of variables and the same set of conditions, but their subsets of functional or augmented variables are not the same; their subsets of forced, augmented or natural conditions are not the same.

(2) Two generalized equivalent variational principles are said to be equivalent if their subsets of functional and augmented variables are the same separately: their subsets of forced, augmented and natural conditions are the same separately.

(3) Two equivalent variational principles are said to be identical if their functionals are identical or will be identical if a proportional factor is considered.

For later convenience of quoting, all the conditions of small displacement theory in elasticity (including basic differential equations and boundary conditions) are listed as follows, where three types of variables (displacement $\mathbf{u}$, strain $\boldsymbol{\varepsilon}$ and stress $\boldsymbol{\sigma}$) are involved:

(1) Differential equilibrium equations:

$$\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}} = \mathbf{0} \qquad \text{(in volume } V\text{).} \tag{1.1}$$

(2) Strain and displacement relations:

$$\boldsymbol{\varepsilon}-\mathbf{D}^T\mathbf{u} = \mathbf{0} \qquad \text{(in } V\text{).} \tag{1.2}$$

(3) Stress and strain relations:

$$\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon} = \mathbf{0} \qquad \text{or} \qquad \boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma} = 0 \qquad \text{(in } V\text{),} \tag{1.3}$$

in which

$$\mathbf{A} = \mathbf{a}^{-1}.$$

(4) Boundary conditions of given displacements:

$$\mathbf{u}-\bar{\mathbf{u}} = \mathbf{0} \qquad \text{(on fixed boundary } S_u\text{).} \tag{1.4}$$

(5) Boundary conditions of given external forces:

$$\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}} = \mathbf{0} \qquad \text{(on free boundary } S_\sigma\text{),} \tag{1.5}$$

in which

$$\mathbf{u} = [u\, v\, w]^T,$$
$$\boldsymbol{\varepsilon} = [\varepsilon_x\, \varepsilon_y\, \varepsilon_z\, \gamma_{yz}\, \gamma_{zx}\, \gamma_{xy}]^T,$$
$$\boldsymbol{\sigma} = [\sigma_x\, \sigma_y\, \sigma_z\, \tau_{yz}\, \tau_{zx}\, \tau_{xy}]^T,$$

$$\mathbf{a} = \frac{1}{E}\begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix}, \tag{1.6}$$

$$\mathbf{D} = \begin{bmatrix} \partial/\partial x & 0 & 0 & 0 & \partial/\partial z & \partial/\partial y \\ 0 & \partial/\partial y & 0 & \partial/\partial z & 0 & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial y & \partial/\partial x & 0 \end{bmatrix}, \tag{1.7}$$

$$\mathbf{L}=\begin{bmatrix} l & 0 & 0 & 0 & n & m \\ 0 & m & 0 & n & 0 & l \\ 0 & 0 & n & m & l & 0 \end{bmatrix}, \tag{1.8}$$

where l, m and n are the directional cosines of outward normal of the boundary.

In elasticity, various variational principles and their functionals have been proposed. Among these functionals the most important ones may be listed as follows:

(1) Conditional and nonconditional potential energy functional $\pi_p(\mathbf{u})$ and $\pi_{1p}(\mathbf{u})$:

$$\pi_p(\mathbf{u})=\int_V [\tfrac{1}{2}(\mathbf{D}^T\mathbf{u})^T\mathbf{A}(\mathbf{D}^T\mathbf{u})-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V-\int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S, \tag{1.9a}$$

$$\pi_{1p}(\mathbf{u})=\int_V [\tfrac{1}{2}(\mathbf{D}^T\mathbf{u})^T\mathbf{A}(\mathbf{D}^T\mathbf{u})-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V$$
$$-\int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S-\int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\mathbf{A}(\mathbf{D}^T\mathbf{u})\,\mathrm{d}S. \tag{1.9b}$$

(2) Complementary energy functional $\pi_c(\boldsymbol{\sigma})$:

$$\pi_c(\boldsymbol{\sigma})=\int_V \tfrac{1}{2}\,\boldsymbol{\sigma}^T\mathbf{a}\boldsymbol{\sigma}\,\mathrm{d}V-\int_{S_u}\bar{\mathbf{u}}^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S. \tag{1.10}$$

(3) Hellinger–Reissner functional[5, 6] $\pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma})$:

$$\pi_{\mathrm{HR}}(\mathbf{u}\boldsymbol{\sigma})=\int_V [\boldsymbol{\sigma}^T\mathbf{D}^T\mathbf{u}-\tfrac{1}{2}\,\boldsymbol{\sigma}^T\mathbf{a}\boldsymbol{\sigma}-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V-\int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S-\int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S. \tag{1.11}$$

(4) Hu–Washizu functional[7, 8] $\pi_{\mathrm{HW}}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$:

$$\pi_{\mathrm{HW}}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})=\int_V [\tfrac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}-\mathbf{D}^T\mathbf{u})-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V-\int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S-\int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S. \tag{1.12}$$

$\pi_p(\mathbf{u})$, $\pi_{1p}(\mathbf{u})$ and $\pi_c(\boldsymbol{\sigma})$ are functionals with single variable, while $\pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma})$ and $\pi_{\mathrm{HW}}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ are functionals with two or three variables, respectively. It should be noted that in all these functionals there is no arbitrary parameter involved.

In this paper, three patterns of functional transformation of variational principles in elasticity are defined and generalized variational principles with several arbitrary parameters are formulated. Henceforth, the following formula of integration by parts will be used:

$$\int_V (\mathbf{D}\boldsymbol{\sigma})^T\mathbf{u}\,\mathrm{d}V=-\int_V \boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u})\,\mathrm{d}V+\int_S(\mathbf{L}\boldsymbol{\sigma})^T\mathbf{u}\,\mathrm{d}S. \tag{1.13}$$

2. TRANSFORMATION PATTERN I OF THE FUNCTIONAL (RELAXATION PATTERN)

The original functional $\pi^{(c)}$ is a conditional one; the forced conditions are

$$\boldsymbol{\phi}_i=\mathbf{0}\qquad(\text{in }\tau_i), \tag{2.1}$$

and the transformed functional $\pi^{(u)}$ is a nonconditional one:

$$\pi^{(u)} = \pi^{(c)} + \sum_i \int_{\tau_i} \lambda_i^T \phi_i \, d\tau_i \tag{I}$$

where λ_i is the multiplier. The functional variables of $\pi^{(u)}$ consist of the functional variables of $\pi^{(c)}$ and multiplier variables. The physical meaning of the multiplier variables can be identified by the use of natural conditions of functional $\pi^{(u)}$.

The feature of pattern I is to transform the conditional functional $\pi^{(c)}$ into nonconditional functional $\pi^{(u)}$. $\pi^{(c)}$ is generalized equivalent but not equivalent to $\pi^{(u)}$. The forced condition (2.1) of $\pi^{(c)}$ is transferred into the natural condition of $\pi^{(u)}$.

3. TRANSFORMATION PATTERN II OF THE FUNCTIONAL (AUGMENTED PATTERN)

The feature of pattern II is that the nonconditional functional $\pi^{(-)}$ which has less variables is transformed into a nonconditional functional $\pi^{(+)}$ which has more variables; the latter is a generalized equivalent to the former one. The augmented conditions of $\pi^{(-)}$ are transferred into the natural conditions of $\pi^{(+)}$.

Here we will explain in detail. Assume that functional $\pi^{(-)}(\mathbf{y})$ is the nonconditional functional before the transformation which has less variables; $\mathbf{y}$ is the functional variable. Assume that $\mathbf{z}$ is the augmented variable. The corresponding augmented conditions are

$$\mathbf{z} - f(\mathbf{y}) = \mathbf{0} \qquad (\text{in } V). \tag{3.1}$$

Then after transformation, the new functional $\pi^{(+)}(\mathbf{y}, \mathbf{z})$ is a nonconditional functional with more variables; $\mathbf{y}$ and $\mathbf{z}$ are the functional variables. It will be determined by the following expression:

$$\pi^{(+)}(\mathbf{y}, \mathbf{z}) = \pi^{(-)}(\mathbf{y}) + \eta Q(\mathbf{y}, \mathbf{z}), \tag{II}$$

in which Q is a positive definite quadratic integral for the augmented condition expression on the left side of eqn (3.1):

$$Q = \int_V \tfrac{1}{2}[\mathbf{z} - f(\mathbf{y})]^T \mathbf{C}[\mathbf{z} - f(\mathbf{y})] \, dV, \tag{3.2}$$

where $\mathbf{C}$ is a positive definite symmetric matrix, and η is an arbitrary nonzero parameter. It can be proved that the original and the transformed functionals $\pi^{(-)}(\mathbf{y})$ and $\pi^{(+)}(\mathbf{y}, \mathbf{z})$ are generalized equivalent to each other. For this reason, we present the relevant theorem of pattern II and its proof as follows:

Theorem. The new functional $\pi^{(+)}(\mathbf{y}, \mathbf{z})$ defined by eqn (II) and the original functional $\pi^{(-)}(\mathbf{y})$ are generalized equivalent to each other. In other words, the stationary conditions of $\pi^{(+)}(\mathbf{y}, \mathbf{z})$,

$$\delta\pi^{(+)}(\mathbf{y}, \mathbf{z}) = 0, \tag{3.3}$$

may be derived from the stationary conditions and augmented conditions of $\pi^{(-)}(\mathbf{y})$:

$$\delta\pi^{(-)}(\mathbf{y}) = 0, \tag{3.4a}$$

$$\mathbf{z} - f(\mathbf{y}) = \mathbf{0}. \tag{3.4b}$$

Conversely, eqns (3.4a, b) can also be derived from eqn (3.3).

Proof. First, we will prove that eqn (3.3) can be derived from eqns (3.4a, b).

The variation of eqn (II) is

$$\delta\pi^{(+)}(\mathbf{y},\mathbf{z}) = \delta\pi^{(-)}(\mathbf{y})+\eta\int_V [\mathbf{z}-f(\mathbf{y})]^T\mathbf{C}\delta[\mathbf{z}-f(\mathbf{y})]\,\mathrm{d}V. \tag{3.5}$$

We have assumed that eqns (3.4a, b) are satisfied; then substituting eqns (3.4a, b) into eqn (3.5), eqn (3.3) can be obtained.

Secondly, it will be proved that eqns (3.4a, b) can be derived from eqn (3.3).

Because both $\delta\mathbf{y}$ and $\delta\mathbf{z}$ are independent variations, both $\delta\mathbf{y}$ and $\delta[\mathbf{z}-f(\mathbf{y})]$ are independent variations also. Since eqn (3.3) is assured, from eqn (3.5) we obtain

$$\delta\pi^{(-)}(\mathbf{y}) = 0, \tag{3.6}$$

$$\mathbf{C}[\mathbf{z}-f(\mathbf{y})] = \mathbf{0}. \tag{3.7}$$

Because $\mathbf{C}$ is a positive definite matrix, then from eqn (3.7) we obtain

$$\mathbf{z}-f(\mathbf{y}) = \mathbf{0}. \tag{3.8}$$

According to eqns (3.6) and (3.8), eqns (3.4a, b) can be proved.

Example 1. It is necessary to derive the augmented functional of Hellinger–Reissner functional $\pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma})$ according to pattern II.

Solution. The original nonconditional functional with less variables is

$$\pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma}) = \int_V [\boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u})-\tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{a}\boldsymbol{\sigma}-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V-\int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S-\int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S, \tag{3.9}$$

the augmented condition of which is

$$\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma} = \mathbf{0} \qquad (\text{in } V). \tag{3.10}$$

According to the transformation pattern II, the positive definite quadratic integral for the augmented condition expression on the left side of eqn (3.10) should be written in the same manner as eqn (3.2):

$$Q = \int_V \tfrac{1}{2}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})^T\mathbf{A}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})\,\mathrm{d}V. \tag{3.11}$$

Substituting into eqn (II), the general form of transformed nonconditional functional with more variables can be obtained as follows:

$$\pi^{(+)}(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma})+\eta\int_V \tfrac{1}{2}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})^T\mathbf{A}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})\,\mathrm{d}V. \tag{3.12}$$

If the parameter η in eqn (3.12) equals to $+1$, special form of augmented functional is obtained:

$$\pi^{(+)}(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon})|_{\eta=+1} = \pi_{\mathrm{HR}}(\mathbf{u},\boldsymbol{\sigma})+\int_V \tfrac{1}{2}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})^T\mathbf{A}(\boldsymbol{\varepsilon}-\mathbf{a}\boldsymbol{\sigma})\,\mathrm{d}V. \tag{3.13}$$

In fact, this functional is the same as the functional $\pi_{\mathrm{HW}}(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ in Hu–Washizu principle.

4. TRANSFORMATION PATTERN III OF THE FUNCTIONAL (EQUIVALENT PATTERN)

The feature of pattern III is to transform the nonconditional functional π into nonconditional functional π_L involving several arbitrary parameters:

$$\pi_L = \pi + \sum_i \eta_i Q_i, \tag{III}$$

in which η_i are arbitrary parameters, and Q_i are the quadratic integrals constituted by natural condition expressions of π in corresponding domains. In general, the new functional π_L is the general form of the functional which is equivalent to the original π. In the case of degeneration (when η_i equals a critical value η_{ci}), $\pi_L|_{\eta_i=\eta_{ci}}$ degenerates into the nonconditional functional with less variables. Some of the functional variables in the original functional are transferred into the augmented variables in the new functional; some of the natural conditions of the original functional are transferred into the augmented conditions of the new functional. Thus the new functional and the original one are generalized equivalent but not equivalent to each other.

Example 2. Derive the equivalent functional $\pi_L(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ of Hu–Washizu functional $\pi_{HW}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ according to pattern III.

Solution. The Hu–Washizu functional is given in eqn (1.12), the natural conditions of which are the equations which define the problems in small displacement theory of elasticity, i.e. eqns (1.1) to (1.5).

As an example, the natural condition (1.3) is transformed according to pattern III. We obtain

$$Q_1 = \int_V \tfrac{1}{2}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})^T\mathbf{a}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})\,\mathrm{d}V, \tag{4.1}$$

$$\pi_{L\eta_1} = \pi_{HW}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})+\eta_1\int_V \tfrac{1}{2}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})^T\mathbf{a}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})\,\mathrm{d}V. \tag{4.2}$$

When η_1 is an arbitrary constant except -1, $\pi_{L\eta_1}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ is the nonconditional functional with three types of variables, and one parameter is involved in the functional.

When $\eta_1 = -1$,

$$\pi_{L\eta_1}|_{\eta_1=-1} = \pi_{HR}(\mathbf{u},\boldsymbol{\sigma}); \tag{4.3}$$

thus the new functional degenerates into the nonconditional functional with two types of variables, $\boldsymbol{\varepsilon}$ degenerates into the augmented variable, and condition (1.3) the augmented condition.

5. GENERALIZED VARIATIONAL PRINCIPLE INVOLVING SEVERAL ARBITRARY PARAMETERS

Let the Hu–Washizu functional $\pi_{HW}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ be the original functional. By use of its five natural conditions (1.1)–(1.5), it is transformed according to pattern III. The more general form of its equivalent functional may be obtained as follows:

$$\pi_L(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = \pi_{HW}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})+\sum_{i=1}^{n}\eta_i Q_i, \tag{5.1}$$

in which Q_i are the quadratic integrals constituted by five natural condition expressions on the left side of eqns (1.1)–(1.5).

According to the three natural condition expressions ψ_1, ψ_2, ψ_3 in the volume domain on the left side of eqns (1.1)–(1.3), six quadratic integrals may be constituted in the following form:

$$\int_V \psi_j^T \mathbf{C}_{jk} \psi_k \, dV \qquad (j, k = 1, 2, 3). \tag{5.2}$$

Choose the matrix **C** properly. The first six quadratic integrals may be obtained as follows:

$$\begin{aligned}
Q_1 &= \int_V \tfrac{1}{2}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})^T \mathbf{a}(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma}) \, dV, \\
Q_2 &= \int_V \tfrac{1}{2}(\mathbf{D}^T\mathbf{u}-\boldsymbol{\varepsilon})^T \mathbf{A}(\mathbf{D}^T\mathbf{u}-\boldsymbol{\varepsilon}) \, dV, \\
Q_3 &= \int_V \tfrac{1}{2}(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}}) \, dV, \\
Q_4 &= \int_V (\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})^T(\boldsymbol{\varepsilon}-\mathbf{D}^T\mathbf{u}) \, dV, \\
Q_5 &= \int_V (\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})^T \mathbf{B}(\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon}) \, dV, \\
Q_6 &= \int_V (\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})^T \mathbf{B}\mathbf{A}(\boldsymbol{\varepsilon}-\mathbf{D}^T\mathbf{u}) \, dV,
\end{aligned} \tag{5.3}$$

in which

$$\mathbf{B} = \begin{bmatrix} \beta_1 & 0 & 0 & 0 & \beta_3 & \beta_2 \\ 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_1 \\ 0 & 0 & \beta_3 & \beta_2 & \beta_1 & 0 \end{bmatrix}.$$

β_1, β_2, β_3 are arbitrary given constants.

By the natural conditions on boundary S_u or S_σ, another eight quadratic integrals are chosen:

$$\begin{aligned}
Q_7 &= \int_{S_u} \tfrac{1}{2}(\mathbf{u}-\bar{\mathbf{u}})^T(\mathbf{u}-\bar{\mathbf{u}}) \, dS, \\
Q_8 &= \int_{S_u} (\mathbf{u}-\bar{\mathbf{u}})^T \mathbf{L}(\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon}) \, dS, \\
Q_9 &= \int_{S_u} (\mathbf{u}-\bar{\mathbf{u}})^T \mathbf{L}\mathbf{A}(\boldsymbol{\varepsilon}-\mathbf{D}^T\mathbf{u}) \, dS, \\
Q_{10} &= \int_{S_u} (\mathbf{u}-\bar{\mathbf{u}})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}}) \, dS, \\
Q_{11} &= \int_{S_\sigma} \tfrac{1}{2}(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}}) \, dS, \\
Q_{12} &= \int_{S_\sigma} (\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T \mathbf{L}(\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon}) \, dS,
\end{aligned}$$

$$Q_{13} = \int_{S_\sigma} (\mathbf{L}\boldsymbol{\sigma} - \bar{\mathbf{T}})^T \mathbf{L}\mathbf{A}(\boldsymbol{\varepsilon} - \mathbf{D}^T\mathbf{u})\,\mathrm{d}S,$$

$$Q_{14} = \int_{S_\sigma} (\mathbf{L}\boldsymbol{\sigma} - \bar{\mathbf{T}})^T (\mathbf{D}\boldsymbol{\sigma} + \bar{\mathbf{F}})\,\mathrm{d}S. \tag{5.4}$$

Substituting eqns (5.3), (5.4) into eqn (5.1), we obtain

$$\begin{aligned}
\pi_L(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) &= \pi_{HW}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) + \sum_{i=1}^{14} \eta_i Q_i \\
&= \int_V \{[-\bar{\mathbf{F}}^T\mathbf{u} + \eta_2 \tfrac{1}{2}(\mathbf{D}^T\mathbf{u})^T\mathbf{A}(\mathbf{D}^T\mathbf{u})] + (1+\eta_1+\eta_2+2\eta_4)\tfrac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{A}\boldsymbol{\varepsilon} \\
&\quad + [\eta_1 \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{a}\boldsymbol{\sigma} + \eta_3 \tfrac{1}{2}(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}}) + \eta_5(\mathbf{B}\boldsymbol{\sigma})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})] \\
&\quad - (\eta_2+\eta_4)(\mathbf{A}\boldsymbol{\varepsilon})^T(\mathbf{D}^T\mathbf{u}) + [(1+\eta_4)\boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u}) - \eta_6(\mathbf{B}\mathbf{A}\mathbf{D}^T\mathbf{u})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})] \\
&\quad + [-(1+\eta_1+\eta_4)\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} - (\eta_5-\eta_6)(\mathbf{B}\mathbf{A}\boldsymbol{\varepsilon})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})]\}\,\mathrm{d}V \\
&\quad + \int_{S_u} \{[\eta_7 \tfrac{1}{2}(\mathbf{u}-\bar{\mathbf{u}})^T(\mathbf{u}-\bar{\mathbf{u}}) - \eta_9(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\mathbf{A}\mathbf{D}^T\mathbf{u}] \\
&\quad - (\eta_8-\eta_9)(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\mathbf{A}\boldsymbol{\varepsilon} - (1-\eta_8)(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma} + \eta_{10}(\mathbf{u}-\bar{\mathbf{u}})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})\}\,\mathrm{d}S \\
&\quad + \int_{S_\sigma} \{-\bar{\mathbf{T}}^T\mathbf{u} + \tfrac{1}{2}(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T[\eta_{11}(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}}) + 2\eta_{12}\mathbf{L}\boldsymbol{\sigma}] - \eta_{13}(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T\mathbf{L}\mathbf{A}\mathbf{D}^T\mathbf{u} \\
&\quad - (\eta_{12}-\eta_{13})(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T\mathbf{L}\mathbf{A}\boldsymbol{\varepsilon} + \eta_{14}(\mathbf{L}\boldsymbol{\sigma}-\bar{\mathbf{T}})^T(\mathbf{D}\boldsymbol{\sigma}+\bar{\mathbf{F}})\}\,\mathrm{d}S.
\end{aligned} \tag{5.5}$$

The functional $\pi_L(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ is equivalent to the Hu–Washizu functional $\pi_{HW}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$. And it is the general form of functional involving arbitrary parameters with three variables $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ except the three degenerate cases listed below.

(1) The functional with two types of variables $\pi_L(\mathbf{u}, \boldsymbol{\sigma})$—the degenerate case in which $\boldsymbol{\varepsilon}$ is excluded.

It is known from eqn (5.5), if $\boldsymbol{\varepsilon}$ is excluded from functional π_L, it may be assumed that

$$\eta_2 = -\eta_4 = 1+\eta_1, \qquad \eta_5 = \eta_6, \qquad \eta_8 = \eta_9, \qquad \eta_{12} = \eta_{13}. \tag{5.6}$$

Then the following nonconditional functional $\pi_L(\mathbf{u}, \boldsymbol{\sigma})$ involving several arbitrary parameters with two types of variables $(\mathbf{u}, \boldsymbol{\sigma})$ may be obtained:

$$\begin{aligned}
\pi_L(\mathbf{u}, \boldsymbol{\sigma}) = \pi_{HR}(\mathbf{u}, \boldsymbol{\sigma}) &+ (1+\eta_1)(Q_1+Q_2-Q_4) + \eta_3 Q_3 + \eta_5(Q_5+Q_6) + \eta_7 Q_7 \\
&+ \eta_8(Q_8+Q_9) + \eta_{10}Q_{10} + \eta_{11}Q_{11} + \eta_{12}(Q_{12}+Q_{13}) + \eta_{14}Q_{14}.
\end{aligned} \tag{5.7}$$

(2) The functional with two types of variables $\pi_L(\mathbf{u}, \boldsymbol{\varepsilon})$—the degenerate case in which $\boldsymbol{\sigma}$ is excluded.

If $\boldsymbol{\sigma}$ is excluded from functional π_L, it may be assumed that

$$\eta_4 = -1, \qquad \eta_8 = 1, \qquad \eta_1 = \eta_3 = \eta_5 = \eta_6 = \eta_{10} = \eta_{11} = \eta_{12} = \eta_{13} = \eta_{14} = 0. \tag{5.8}$$

Then the following nonconditional functional $\pi_L(\mathbf{u}, \boldsymbol{\varepsilon})$ involving several arbitrary parameters with two types of variables $(\mathbf{u}, \boldsymbol{\varepsilon})$ may be obtained:

$$\pi_L(\mathbf{u}, \boldsymbol{\varepsilon}) = (\pi_{HW} - Q_4 + Q_8) + \eta_2 Q_2 + \eta_7 Q_7 + \eta_9 Q_9. \tag{5.9}$$

(3) The functional with single type of variable $\pi_L(\mathbf{u})$—the degenerate case in which $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are excluded.

In functional π_L, if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are excluded, it may be assumed in eqn (5.5) that

$$\begin{gathered} \eta_2 = \eta_8 = \eta_9 = 1, \qquad \eta_4 = -1, \\ \eta_1 = \eta_3 = \eta_5 = \eta_6 = \eta_{10} = \eta_{11} = \eta_{12} = \eta_{13} = \eta_{14} = 0. \end{gathered} \tag{5.10}$$

Then the nonconditional functional $\pi_L(\mathbf{u})$ involving arbitrary parameters with single type of variable $\mathbf{u}$ may be obtained as follows:

$$\pi_L(\mathbf{u}) = (\pi_{HW} + Q_2 - Q_4 + Q_8 + Q_9) + \eta_7 Q_7 = \pi_{1p}(\mathbf{u}) + \eta_7 Q_7. \tag{5.11}$$

(5.5), (5.7), (5.9) and (5.11) are nonconditional functionals involving arbitrary parameters with three variables ($\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$), two variables ($\mathbf{u}$, $\boldsymbol{\sigma}$), ($\mathbf{u}$, $\boldsymbol{\varepsilon}$) and single variable ($\mathbf{u}$), respectively.

6. SIMPLE FORMS OF FUNCTIONALS INVOLVING ARBITRARY PARAMETERS

For the convenience of application, we consider simple forms of functionals involving arbitrary parameters.

(1) Simple form of functional with three variables ($\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$) involving three arbitrary parameters.

Assume that in functional (5.5) all the parameters except η_1, η_2, η_4 equal to zero; the following functional can be obtained:

$$\pi_{L1}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \pi_{HW} + \eta_1 Q_1 + \eta_2 Q_2 + \eta_4 Q_4. \tag{6.1}$$

Equation (6.1) is a simple form of functional involving three arbitrary parameters with three variables ($\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$). The following cases:

$$\begin{aligned} &\text{(i)} \quad 1 + \eta_1 + \eta_4 = 0 \quad \text{and} \quad \eta_2 + \eta_4 = 0, \\ &\text{(ii)} \quad \eta_1 = 0 \quad \text{and} \quad \eta_4 = -1, \end{aligned} \tag{6.2}$$

are degenerate cases of (6.1).

When different values of η_1, η_2, η_4 are chosen, different special cases of functional (6.1) can be obtained.

(a) When $\eta_1 = \eta_2 = \eta_4 = 0$, the Hu–Washizu functional $\pi_{HW}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ is obtained.

(b) When $\eta_2 = \eta_4 = 0$, the functional proposed by Chien[9] is obtained:

$$\pi_{Ch}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_V \left[\boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u}) + \frac{1+\eta_1}{2} \boldsymbol{\varepsilon}^T \mathbf{A} \boldsymbol{\varepsilon} + \tfrac{1}{2}\eta_1 \boldsymbol{\sigma}^T \mathbf{a} \boldsymbol{\sigma} - (1+\eta_1)\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} - \bar{\mathbf{F}}^T \mathbf{u} \right] \mathrm{d}V - \int_{S_u} (\mathbf{u} - \bar{\mathbf{u}})^T \mathbf{L} \boldsymbol{\sigma} \, \mathrm{d}S - \int_{S_\sigma} \bar{\mathbf{T}}^T \mathbf{u} \, \mathrm{d}S, \tag{6.3}$$

in which $\eta_1 = -1$ implies a degenerate case.

(c) When $\eta_1 = 1$, $\eta_2 = 0$, $\eta_4 = -1$ or $\eta_1 = \eta_2 = 1$, $\eta_4 = -1$, the following functionals are obtained, respectively:

$$\pi_{L2}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_V [\boldsymbol{\varepsilon}^T(\mathbf{A}\mathbf{D}^T\mathbf{u} - \boldsymbol{\sigma}) + \tfrac{1}{2}\boldsymbol{\sigma}^T \mathbf{a} \boldsymbol{\sigma} - \bar{\mathbf{F}}^T \mathbf{u}] \, \mathrm{d}V - \int_{S_u} (\mathbf{u} - \bar{\mathbf{u}})^T \mathbf{L} \boldsymbol{\sigma} \, \mathrm{d}S - \int_{S_\sigma} \bar{\mathbf{T}}^T \mathbf{u} \, \mathrm{d}S, \tag{6.4}$$

$$\pi_{L3}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = \int_V [\tfrac{1}{2}(\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon})^T\mathbf{a}(\boldsymbol{\sigma}-\mathbf{A}\boldsymbol{\varepsilon})+\tfrac{1}{2}(\mathbf{D}^T\mathbf{u})^T\mathbf{A}(\mathbf{D}^T\mathbf{u})-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V - \int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S - \int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S. \quad (6.5)$$

Both functionals $\pi_{L2}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ and $\pi_{L3}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ are as simple as $\pi_{HW}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ and are equivalent to it.

(2) Simple form of functional with two variables $(\mathbf{u},\boldsymbol{\sigma})$ involving one arbitrary parameter.

Assume that in functional (5.7) all the parameters except η_1 are equal to zero; a simple form of functional involving one arbitrary parameter with two variables $(\mathbf{u},\boldsymbol{\sigma})$ is obtained:

$$\pi_{L1}(\mathbf{u},\boldsymbol{\sigma}) = \pi_{HR}(\mathbf{u},\boldsymbol{\sigma})+(1+\eta_1)\int_V \tfrac{1}{2}(\mathbf{D}^T\mathbf{u}-\mathbf{a}\boldsymbol{\sigma})^T\mathbf{A}(\mathbf{D}^T\mathbf{u}-\mathbf{a}\boldsymbol{\sigma})\,\mathrm{d}V. \quad (6.6)$$

Two special cases of functional (6.6) are given below:

(a) When $\eta_1 = -1$, the Hellinger–Reissner functional $\pi_{HR}(\mathbf{u},\boldsymbol{\sigma})$ is obtained.

(b) When $\eta_1 = 1$, we have

$$\pi_{L2}(\mathbf{u},\boldsymbol{\sigma}) = \int_V [(\mathbf{D}^T\mathbf{u})^T\mathbf{A}(\mathbf{D}^T\mathbf{u})+\tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{a}\boldsymbol{\sigma}-\boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u})-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V - \int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S - \int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\boldsymbol{\sigma}\,\mathrm{d}S. \quad (6.7)$$

If the boundary condition $\mathbf{u}-\bar{\mathbf{u}} = \mathbf{0}$ (on S_u) is satisfied, then the last term of (6.7) vanishes and the functional $M(\mathbf{u},\boldsymbol{\sigma})$ given by Oden[10] is obtained.

(3) Simple form of functional with two variables $(\mathbf{u},\boldsymbol{\varepsilon})$ involving one arbitrary parameter.

Assume that in functional (5.9) all the parameters except η_2 are equal to zero; a simple form of functional involving one arbitrary parameter with two variables $(\mathbf{u},\boldsymbol{\varepsilon})$ is obtained:

$$\pi_{L1}(\mathbf{u},\boldsymbol{\varepsilon}) = (\pi_{HW}-Q_4+Q_8)+\eta_2 Q_2. \quad (6.8)$$

When $\eta_2 = 0$, a special case of (6.8) is obtained:

$$\pi_{L2}(\mathbf{u},\boldsymbol{\varepsilon}) = \pi_{HW}-Q_4+Q_8 = \int_V [\boldsymbol{\varepsilon}^T\mathbf{A}(\mathbf{D}^T\mathbf{u})-\tfrac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{A}\boldsymbol{\varepsilon}-\bar{\mathbf{F}}^T\mathbf{u}]\,\mathrm{d}V - \int_{S_\sigma}\bar{\mathbf{T}}^T\mathbf{u}\,\mathrm{d}S - \int_{S_u}(\mathbf{u}-\bar{\mathbf{u}})^T\mathbf{L}\mathbf{A}\boldsymbol{\varepsilon}\,\mathrm{d}S. \quad (6.9)$$

If $\mathbf{A}\boldsymbol{\varepsilon}$ in the last integral is replaced by $\boldsymbol{\sigma}$, then the functional $R_\varepsilon(\mathbf{u},\boldsymbol{\varepsilon})$ given in [10] is obtained. It seems that $R_\varepsilon(\mathbf{u},\boldsymbol{\varepsilon})$ is not a functional with two variables $(\mathbf{u},\boldsymbol{\varepsilon})$.

(4) Simple form of functional with single variable $(\mathbf{u})$ involving one arbitrary parameter.

Functional (5.11) is a simple form of functional with single variable $(\mathbf{u})$ involving an arbitrary parameter η_7. When $\eta_7 = 0$, the nonconditional potential energy functional $\pi_{1p}(\mathbf{u})$ given in (1.9b) is obtained as a special case of functional (5.11).

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